

The functional definition of geodesics

Lubomír Klapka

Abstract. In this paper a functional definition of geodesics is introduced which allows to generalize the notion of a geodesic from smooth to topological manifolds. It is shown that in the C^∞ -case the new definition coincides with the classical definition of geodesics of a linear connection. If C^∞ -smoothness is not required, it is shown by an example that the new definition includes geodesics of non-linear, homogeneous connections. Moreover, an example of generalized geodesics which does not arise from any connection is presented here.

1 Introduction

The concept of a geodesic is usually introduced by means of a linear connection which has been developed from the Levi-Civita parallelism in Riemannian geometry. The general geometric theory of linear and non-linear connections, as applied in this paper, can be found in [7]. In this approach, a geodesic is always a solution of a second order differential equation. In Sec. 2 we are recalling the well-known definition of a linear connection based on the chart expression of this differential equation, and some relations between geodesics and geodesic arcs.

Sec. 3 is devoted to open convex sets in a smooth manifold. Using the Whitehead lemma [9], we prove the existence of a covering of a smooth manifold by open sets whose intersections are convex. We also derive here functional equations (5), (6) for geodesic arcs on a convex manifold.

In Sec. 4, these functional equations are generalized to manifolds which are not necessarily convex and a new definition of geodesics is introduced. Since the equations do not include derivatives, the new definition makes sense on any topological manifold. We also give here two important properties of the solutions of the functional equations.

In Sec. 5, we prove that in the C^∞ -case, the functional definition of a geodesic coincides with the usual one. The proof in one direction uses the

covering constructed in Sec. 3. The proof of the converse is based on the author's ideas contained in [4].

In Sec. 6, we discuss some solutions of (5) and (6) which are obtained without the smoothness assumption. Since the general solution is not known, a few particular examples are considered: a linear solution, a continuous solution, a solution equivalent to a non-linear connection, and a solution which does not correspond with any connection.

By a manifold we mean in this paper a finite-dimensional manifold of class C^k , where $k \in \{0, 1, 2, \dots, \infty\}$; topologically we assume a manifold to be second countable and Hausdorff. Manifolds with a boundary are also used. For more information, the reader is referred to [5] and [6].

Throughout this paper we will use the notation $\frac{\partial f}{\partial x^i}$ to mean $(f \circ x^{-1})_i \circ x$, where f is a C^1 -mapping, $\text{codom } f$ is a real finite-dimensional vector space, x^i is the i th coordinate function of a coordinate system x on $\text{dom } f$, and $(f \circ x^{-1})_i$ is the partial derivative of $f \circ x^{-1}$ with respect to the i th variable.

2 The usual definition of geodesics

Let M be a C^2 -manifold. A *linear connection* Γ on M is a set of functions $\Gamma_{jk}^i : \text{dom } x \rightarrow \mathbb{R}$, $i, j, k \in \{1, 2, \dots, \dim M\}$, satisfying

$$\frac{\partial^2 \bar{x}^i}{\partial x^j \partial x^k} + \bar{\Gamma}_{lm}^i \frac{\partial \bar{x}^l}{\partial x^j} \frac{\partial \bar{x}^m}{\partial x^k} = \frac{\partial \bar{x}^i}{\partial x^l} \Gamma_{jk}^l \quad (1)$$

on $\text{dom } x \cap \text{dom } \bar{x}$, where x and \bar{x} runs over all coordinate systems on M . The functions $\Gamma_{jk}^i : \text{dom } x \rightarrow \mathbb{R}$ are called *components* of Γ with respect to x . The manifold M is called a *domain of* Γ and is denoted by $\text{dom } \Gamma$. A linear connection Γ is said to be *smooth* if $\text{dom } \Gamma$ is C^∞ -manifold and all components of Γ are C^∞ -functions.

Definition 1 Let Γ be a linear connection, $I \subset \mathbb{R}$ an open interval. A C^2 -map $g : I \rightarrow \text{dom } \Gamma$ is called a *geodesic of* Γ if for any $\tau \in I$ there exist coordinate systems t on I and x on $\text{dom } \Gamma$ such that $\tau \in \text{dom } t$, $t = \text{id}_{\text{dom } t}$, $g(\text{dom } t) \subset \text{dom } x$, and

$$\frac{\partial^2 x^i \circ g}{\partial t^2} + (\Gamma_{jk}^i \circ g) \frac{\partial x^j \circ g}{\partial t} \frac{\partial x^k \circ g}{\partial t} = 0, \quad (2)$$

where Γ_{jk}^i are components of Γ with respect to x . We denote the set of all geodesics of Γ by $\text{Geo } \Gamma$.

A map $h : [0, 1] \rightarrow \text{dom } \Gamma$ is called a *geodesic arc* of Γ if there exists a geodesic $g \in \text{Geo } \Gamma$ such that $[0, 1] \subset \text{dom } g$, $h = g|_{[0, 1]}$. We denote the set of all geodesic arcs of Γ by $\text{Arc } \Gamma$.

Proposition 1 *Suppose Γ is a linear connection on M , $I \subset \mathbb{R}$ is an open interval, $g : I \rightarrow M$ is a C^2 -map. Then the following three statements are equivalent:*

1. *g is a geodesic of Γ .*
2. *For every two real numbers $\alpha, \beta \in I$, the map*

$$h : [0, 1] \ni \gamma \rightarrow g((1 - \gamma)\alpha + \gamma\beta) \in M \quad (3)$$

is a geodesic arc of Γ .

3. *For every real number $\tau \in I$ there exist real numbers $\alpha, \beta \in I$ such that $\alpha < \tau < \beta$ and (3) is a geodesic arc of Γ .*

Proof. Suppose the statement 1 holds. From Definition 1 it follows that $g \circ p|_{p^{-1}(I)} \in \text{Geo } \Gamma$, where $p : \mathbb{R} \ni \tau \rightarrow ((1 - \tau)\alpha + \tau\beta) \in \mathbb{R}$. Since $h = g \circ p|_{p^{-1}(I)}|_{[0, 1]}$, we obtain the statement 2.

Suppose the statement 2 holds. Since the interval I is open, there must exist real numbers α, β such that the statement 3 holds.

Suppose the statement 3 holds; then g satisfies differential equation (2) on an open neighborhood (α, β) of τ . Since $\tau \in I$ is arbitrary, g satisfies this equation on I . Now, from Definition 1 it follows that the statement 1 holds.

This completes the proof.

3 Convex sets

Let Γ be a linear connection. An open set $U \subset \text{dom } \Gamma$ is *convex* with respect to $\text{Arc } \Gamma$ if for any two points $a, b \in U$ there exists a unique geodesic arc $h \in \text{Arc } \Gamma$ such that $h(0) = a$, $h(1) = b$, $h([0, 1]) \subset U$. Let us denote this geodesic arc by h_{ab} . An open set $U \subset \text{dom } \Gamma$ is *smoothly convex* with respect to $\text{Arc } \Gamma$ if U is convex with respect to $\text{Arc } \Gamma$ and the map

$$f : U \times U \times [0, 1] \ni (a, b, \tau) \rightarrow h_{ab}(\tau) \in U \quad (4)$$

is smooth. It is clear that any open convex subset of a smoothly convex set is smoothly convex. The empty set is, by definition, convex and smoothly

convex. The existence of smoothly convex neighborhoods was proved by J. H. C. Whitehead [9] in 1931. The proof of following lemma can be also found in [3] and [5].

Lemma 1 (Whitehead) *Suppose Γ is a smooth linear connection on a manifold M , $W \subset M$ is an open set, $a \in W$ is a point; then there exists an open set $V \subset M$ such that $a \in V \subset W$, and the set V is smoothly convex with respect to $\text{Arc } \Gamma$.*

The following proposition is needed for the sequel. Its proof is trivial.

Proposition 2 *Suppose Γ is a smooth linear connection, an open set $U \subset \text{dom } \Gamma$ is convex with respect to $\text{Arc } \Gamma$, $h, h' \in \text{Arc } \Gamma$ are geodesic arcs such that $h([0, 1]) \subset U$, $h'([0, 1]) \subset U$, $h(0) = h'(0)$, $h(1) = h'(1)$. Then $h = h'$.*

We note that convex sets in a Euclidean space have different properties than convex sets in a manifold endowed with a linear connection. In particular, the intersection of two convex sets of a Euclidean space is always convex, while the intersection of two convex sets in the circle is not necessarily convex, and even may be non-connected. Whitehead's lemma guarantees existence of a covering by open, convex sets; nothing is claimed, however, about their intersections. In what follows, we need the following result.

Lemma 2 *If Γ is a smooth linear connection on manifold M , then there exists an open covering $\{U_a\}_{a \in M}$ of M , where U_a 's are neighborhoods of a 's, such that the set $U_a \cap U_b$ is smoothly convex with respect to $\text{Arc } \Gamma$ for every two points $a, b \in M$.*

Proof. Consider a smooth Riemannian metric on M . Let $a \in M$ be a point, x a coordinate system on M such that $a \in \text{dom } x$. Let us consider the function $u : \text{dom } x \ni b \rightarrow \Delta^2(a, b) \in \mathbb{R}$, where $\Delta(a, b)$ is the distance of b from a . At the point a , the first partial derivatives of u vanish, and the second partial derivatives are components of the metric tensor. This has been proved by J. L. Synge in [8]. Therefore, the second covariant derivatives $\partial^2 u / \partial x^k \partial x^l - \Gamma_{kl}^m \partial u / \partial x^m$ at a are coefficients in a positive definite quadratic form. From the continuity it follows that there exists an open set W such that $a \in W \subset \text{dom } x$ and the second covariant derivatives of u are coefficients in a positive definite quadratic form at each point $b \in W$. We restrict x to W .

By Lemma 1, there exists an open set V such that $a \in V \subset \text{dom } x$ and V is smoothly convex with respect to $\text{Arc } \Gamma$. Further, there exists an open ball $B_a \subset V$ of center $a \in B_a$ and radius ρ_a . Let us consider two points $c, d \in B_a$. Since V is convex, there is a unique geodesic arc $h \in \text{Arc } \Gamma$ such that $h(0) = c$, $h(1) = d$, and $h([0, 1]) \subset V$. From (2), we get

$$\frac{\partial^2 u \circ h}{\partial t^2} = \left(\left(\frac{\partial^2 u}{\partial x^k \partial x^l} - \Gamma_{kl}^m \frac{\partial u}{\partial x^m} \right) \circ h \right) \frac{\partial x^k \circ h}{\partial t} \frac{\partial x^l \circ h}{\partial t}.$$

Since the quadratic form in the first derivatives is positive definite, we have $\partial^2 u \circ h / \partial t^2 \geq 0$, whence the function $u \circ h$ is convex. In this case, for each $\tau \in [0, 1]$, we have

$$u \circ h(\tau) \leq (1 - \tau)u \circ h(0) + \tau u \circ h(1) < (1 - \tau)\rho_a^2 + \tau\rho_a^2 = \rho_a^2.$$

This means that $h([0, 1]) \subset B_a$. Thus the open set $B_a \subset M$ is smoothly convex with respect to $\text{Arc } \Gamma$. For the same reason, any open ball of center a and radius smaller than ρ_a is smoothly convex with respect to $\text{Arc } \Gamma$.

Thus, there exists an open covering $\{B_a\}_{a \in M}$ of M , where any B_a is a smoothly convex open ball of center a and radius ρ_a . Moreover, there exists an open covering $\{U_a\}_{a \in M}$, where any U_a is the smoothly convex open ball of center a and radius $\frac{1}{3}\rho_a$.

Now, consider two points $a, b \in M$. For the sake of being definite assume that $\rho_a \leq \rho_b$. If $U_a \cap U_b = \emptyset$, then the open set $U_a \cap U_b$ is smoothly convex with respect to $\text{Arc } \Gamma$. Conversely, suppose that $U_a \cap U_b \neq \emptyset$. Consider two points $c, d \in U_a \cap U_b$. Since U_a, U_b are convex sets, there exist geodesic arcs $h, h' \in \text{Arc } \Gamma$ such that $h(0) = h'(0) = c$, $h(1) = h'(1) = d$, $h([0, 1]) \subset U_a$, and $h'([0, 1]) \subset U_b$. From $\frac{2}{3}\rho_a + \frac{1}{3}\rho_b \leq \rho_b$ it follows that $U_a \cup U_b \subset B_b$. Since B_b is a convex set, by Proposition 2, $h = h'$. Hence, we get $h([0, 1]) = h'([0, 1]) \subset U_a \cap U_b$. From here, the open set $U_a \cap U_b$ is, by definition, smoothly convex with respect to $\text{Arc } \Gamma$.

This completes the proof of Lemma 2.

In the following proposition, we consider functional properties of the mapping (4).

Proposition 3 *Suppose Γ is a linear connection on a manifold M , $U \subset M$ is an open set, convex with respect to $\text{Arc } \Gamma$, f is the mapping (4), $a, b \in U$ are any points, and $\alpha, \beta, \gamma \in [0, 1]$ are real numbers. Then*

$$f(a, b, 0) = a, \quad f(a, b, 1) = b, \quad (5)$$

$$f(a, b, (1 - \gamma)\alpha + \gamma\beta) = f(f(a, b, \alpha), f(a, b, \beta), \gamma). \quad (6)$$

Proof. By construction, f satisfies (5), $[0, 1] \ni \tau \rightarrow f(a, b, \tau) \in M$ is a geodesic arc of Γ . From Proposition 1, the mappings $h : [0, 1] \ni \tau \rightarrow f(a, b, (1 - \tau)\alpha + \tau\beta) \in M$, $h' : [0, 1] \ni \tau \rightarrow f(f(a, b, \alpha), f(a, b, \beta), \tau) \in M$ are geodesic arcs of Γ . Combining (5) and Proposition 2, we obtain (6).

This completes the proof.

4 The new definition of geodesics

Let M be a finite-dimensional topological manifold. In this section, we generalize equations (5) and (6) from convex smooth manifolds to non-convex topological manifolds. For this purpose, we replace the smooth mapping (4) by a continuous mapping

$$f : D \times [0, 1] \rightarrow M, \quad (7)$$

where $D \subset M \times M$ is an open set containing the diagonal, and (5), (6) hold for each $(a, b) \in D$, $\alpha, \beta, \gamma \in [0, 1]$.

By Aczél definitions [1], the formulas (5), (6) are called *functional equations*, and a mapping (7) is called a *solution of functional equations*. A solution of functional equation is said to be *smooth* if (7) is C^∞ .

Definition 2 A mapping $g : I \rightarrow M$, where $I \subset \mathbb{R}$ is an open interval, is called a *geodesic*, associated with a solution (7) of the functional equations (5), (6) if and only if to each $\tau \in I$ there exist $\alpha, \beta \in I$, $\alpha < \tau < \beta$, such that for each $\gamma \in [0, 1]$,

$$g((1 - \gamma)\alpha + \gamma\beta) = f(g(\alpha), g(\beta), \gamma). \quad (8)$$

The set of all geodesics associated with f is denoted by $\text{Geo } f$.

An easy consequence of the definition is that geodesics associated with a solution of the functional equations are always continuous.

Before going on, we wish to mention two special cases. Let f be a solution of (5) and (6), let $(a, b, \gamma) \in \text{dom } f$. Taking $\alpha = \beta = 0$ in (6) we get by (5)

$$f(a, a, \gamma) = a. \quad (9)$$

Taking $\alpha = 1$, $\beta = 0$, we obtain

$$f(a, b, 1 - \gamma) = f(b, a, \gamma). \quad (10)$$

5 The comparison of the definitions in C^∞ -case

In this section, we wish to prove that in the C^∞ -case, the new definition of geodesics coincides with the usual one.

Theorem 1 *To every smooth linear connection Γ there exists a smooth solution f of the functional equations (5) and (6) such that $\text{Geo } f = \text{Geo } \Gamma$.*

Proof. 1. First, we construct a solution f . For this purpose we consider a covering $\{U_a\}_{a \in M}$ of $M = \text{dom } \Gamma$ with the properties described by Lemma 2. To each U_a we assign a smooth solution $f_a : U_a \times U_a \times [0, 1] \rightarrow U_a$ of (5), (6) (see Proposition 3).

Let $a, b \in M$, $c, d \in U_a \cap U_b$. Since $U_a \cap U_b$ is convex, there exists just one geodesic arc $h \in \text{Arc } \Gamma$ such that $h(0) = c$, $h(1) = d$, $h([0, 1]) \subset U_a \cap U_b$. Since U_a, U_b are convex, we get for all $\gamma \in [0, 1]$ the equality $h(\gamma) = f_a(c, d, \gamma) = f_b(c, d, \gamma)$. If $c, d \in U_a \cap U_b$ are arbitrary, then $f_a|_{\text{dom } f_a \cap \text{dom } f_b} = f_b|_{\text{dom } f_a \cap \text{dom } f_b}$. Moreover, since $a, b \in M$ are arbitrary, there should exist a mapping $f : D \times [0, 1] \rightarrow M$, where

$$D = \bigcup_{a \in M} U_a \times U_a \subset M \times M,$$

such that for each $a \in M$, $f|_{\text{dom } f_a} = f_a$. The set D is open and contains the diagonal of $M \times M$. Since f_a are smooth solutions of (5), (6), f is also a smooth solution of (5), (6).

2. Now we prove that $\text{Geo } f = \text{Geo } \Gamma$. Let $g \in \text{Geo } f$. Then by Definition 2, to each $\tau \in \text{dom } g$ there exist $\alpha, \beta \in \text{dom } g$ such that relations $\alpha < \tau < \beta$ and (8) hold. The equality (8) can be written as $h = h'$, where h is defined by (3), and h' by

$$h' : [0, 1] \ni \gamma \rightarrow f(g(\alpha), g(\beta), \gamma) \in M. \quad (11)$$

The construction of f implies $h' \in \text{Arc } \Gamma$, and $\tau \in \text{dom } g$ is arbitrary. Thus, Proposition 1 implies $g \in \text{Geo } \Gamma$.

Conversely, assume that $g \in \text{Geo } \Gamma$, $\tau \in \text{dom } g$, $a = g(\tau)$. Since g is continuous, there exist $\alpha, \beta \in \text{dom } g$ for which $\alpha < \tau < \beta$, $g([\alpha, \beta]) \subset U_a$. By proposition 1, $h \in \text{Arc } \Gamma$, and by construction of f , $h' \in \text{Arc } \Gamma$. Using convexity of U_a , (5) and Proposition 2, we get the relation (8). Since $\tau \in \text{dom } g$ is arbitrary, Definition 2 implies $g \in \text{Geo } f$.

This completes the proof.

Theorem 2 *To every smooth solution f of the functional equations (5) and (6) there exists a smooth linear connection Γ such that $\text{Geo } f = \text{Geo } \Gamma$.*

Proof. 1. Let us construct Γ . Let $f : D \times [0, 1] \rightarrow M$ be a smooth solution of (5) and (6), $a \in M$ an arbitrary point, x a coordinate system at a such that $\text{dom } x \times \text{dom } x \subset D$. Let $n = \dim M$. Denote

$$\Gamma_{jk}^i(a) = -4 \frac{\partial^2 x^i \circ f}{\partial y^j \partial z^k} (a, a, \frac{1}{2}), \quad (12)$$

where $y^1, y^2, \dots, y^n, z^1, z^2, \dots, z^n, t$ are the coordinate functions of the coordinate system $x \times x \times \text{id}_{[0,1]}$ on $D \times [0, 1]$ (compare with [4]). By (9) and (10),

$$\frac{\partial x^i \circ f}{\partial y^j} (a, a, \frac{1}{2}) = \frac{\partial x^i \circ f}{\partial z^j} (a, a, \frac{1}{2}) = \frac{1}{2} \delta_j^i, \quad (13)$$

where δ_j^i is the Kronecker symbol. Since $a \in M$ is arbitrary, (9), (12), and (13) imply (1). Thus, (12) defines a linear connection Γ on M . Since the components of Γ are restrictions of the second derivatives of smooth functions to a smooth submanifold, Γ must be smooth.

2. Now we prove that $\text{Geo } f = \text{Geo } \Gamma$. Let $g \in \text{Geo } f$, $\tau \in \text{dom } g$. Then by Definition 2 there exist $\alpha_0, \beta_0 \in \text{dom } g$ such that for $\alpha = \alpha_0$, $\beta = \beta_0$, and $\gamma \in [0, 1]$, relations $\alpha < \tau < \beta$ and (8) hold. Then it follows for $\alpha, \beta \in [\alpha_0, \beta_0]$, $\gamma \in [0, 1]$ that

$$g((1 - \gamma)\alpha + \gamma\beta) = f\left(g(\alpha_0), g(\beta_0), \frac{(1 - \gamma)(\alpha - \alpha_0) + \gamma(\beta - \alpha_0)}{\beta_0 - \alpha_0}\right).$$

But then by (6), the relation (8) holds for any $\alpha, \beta \in [\alpha_0, \beta_0]$, $\gamma \in [0, 1]$. Consider coordinate systems t on $\text{dom } g$ and x on M such that $\tau \in \text{dom } t$, $t = \text{id}_{\text{dom } t}$, $g(\text{dom } t) \subset \text{dom } x$, and $\text{dom } x \times \text{dom } x \subset D$. Further, consider the chart $x \times x \times \text{id}_{[0,1]}$ on $D \times [0, 1]$. Differentiating the coordinate expression of (8) with respect to α and β and setting $\alpha = \beta = \tau$, $\gamma = \frac{1}{2}$, we find, using (12), that g satisfies (2) at the point τ . According to (1), the equation (2) is satisfied also for $\text{dom } x \times \text{dom } x \not\subset D$. Since $\tau \in \text{dom } g$ is arbitrary, and the charts t, x , such that $\tau \in \text{dom } t$, $t = \text{id}_{\text{dom } t}$, $g(\text{dom } t) \subset \text{dom } x$, are also arbitrary, we get, using Definition 1, that g is a geodesic of the connection Γ , i.e., $g \in \text{Geo } \Gamma$.

To prove the converse, choose $g \in \text{Geo } \Gamma$, $\tau \in \text{dom } g$. By Lemma 1, there exists a neighborhood $U \subset M$ of the point $g(\tau)$, convex with respect to $\text{Arc } \Gamma$. Then by (9), the continuity of f , and the compactness of the interval $[0, 1]$,

imply existence of an open set $W \subset M$ such that $g(\tau) \in W$, $W \times W \times [0, 1] \subset \text{dom } f$, and $f(W \times W \times [0, 1]) \subset U$. Consider the mappings h and h' defined by (3) and (11). Since g is continuous, the numbers $\alpha, \beta \in \text{dom } g$ can be chosen in such a way that $\alpha < \tau < \beta$ and $h([0, 1]) \subset U \cap W$. Then indeed $h'([0, 1]) \subset U$. By Proposition 1, $h \in \text{Arc } \Gamma$, and by (6) and Definition 2, $h'|_{(0,1)} \in \text{Geo } f$. But we already know that $\text{Geo } f \subset \text{Geo } \Gamma$, hence $h'|_{(0,1)}$ is a solution of (2). Since f is smooth, the domain of the definition of this solution can be extended to a neighborhood of $[0, 1]$, hence $h' \in \text{Arc } \Gamma$. The convexity of U implies, by (5) and Proposition 2, that (8) holds. Now since $\tau \in \text{dom } g$ is arbitrary, we have, using Definition 2, $g \in \text{Geo } f$, as required.

This completes the proof.

6 Examples

The following examples are obtained by a reparametrization of the geodesic arcs corresponding with the linear solution of functional equations.

Example 1 (linear solution). Let \mathbb{V} be a real, finite-dimensional vector space. We look for a continuous solution $f : \mathbb{V} \times \mathbb{V} \times [0, 1] \rightarrow \mathbb{V}$ of (5) and (6) such that for each $\tau \in [0, 1]$, the mapping $\mathbb{V} \times \mathbb{V} \ni (a, b) \rightarrow f(a, b, \tau) \in \mathbb{V}$ is linear.

By linearity, there exists a mapping $q : [0, 1] \rightarrow \text{Lin}(\mathbb{V}, \mathbb{V})$, assigning to every $\gamma \in [0, 1]$ a linear mapping $q(\gamma) : \mathbb{V} \ni b \rightarrow f(0, b, \gamma) \in \mathbb{V}$. By (5),

$$q(0) = 0. \quad (14)$$

Since for every $a, b \in \mathbb{V}$, $\gamma \in [0, 1]$ we get with help of (10) $f(a, b, \gamma) = q(1 - \gamma)(a) + q(\gamma)(b)$, it follows from (9) that

$$q\left(\frac{1}{2}\right) = \frac{1}{2} \text{id}_{\mathbb{V}}. \quad (15)$$

Substituting into (6) $a = 0$, $\gamma = \frac{1}{2}$, we get, using (10) and (15) the Jensen's functional equation

$$q\left(\frac{\alpha + \beta}{2}\right) = \frac{q(\alpha) + q(\beta)}{2}.$$

Under conditions (14) and (15), this equation has a unique continuous solution $q(\gamma) = \gamma \text{id}_{\mathbb{V}}$ (see e.g. [1]). Thus,

$$f(a, b, \gamma) = (1 - \gamma)a + \gamma b,$$

which satisfies (5) and (6).

It is easily seen that all geodesics associated with this solution are also geodesics of the canonical linear connection on the manifold \mathbb{V} .

Example 2 (continuous solution). Let \mathbb{E} be a real, finite-dimensional vector space, $\mathbb{E} \times \mathbb{E} \ni (a, b) \rightarrow (ab) \in \mathbb{R}$ a scalar product, $\mathbb{E} \ni a \rightarrow |a| \in [0, \infty)$ the corresponding norm, and let $v : \mathbb{R} \rightarrow \mathbb{R}$ be a homeomorphism such that for each $\alpha \in \mathbb{R}$, $v(-\alpha) = -v(\alpha)$. We shall show that the formula

$$f(a, b, \gamma) = \begin{cases} v_{ab}^{-1}((1 - \gamma)v_{ab}(a) + \gamma v_{ab}(b)) & \text{for } a \neq b \\ a & \text{for } a = b, \end{cases} \quad (16)$$

where $v_{ab} : \mathbb{E} \rightarrow \mathbb{E}$ is defined by $v_{ab}(c) = c + (v((ce)) - (ce))e$, $e = (a - b)/|a - b|$, defines a continuous solution $f : \mathbb{E} \times \mathbb{E} \times [0, 1] \rightarrow \mathbb{E}$ of (5), (6).

It is easy to verify that if $a \neq b$, then v_{ab} is a bijection, and $v_{ab}^{-1}(c) = c + (v^{-1}((ce)) - (ce))e$. This shows the existence of (16). If $a = b$, or $a \neq b$ and $\alpha = \beta$, f obviously satisfies (5) and (6). Consider the remaining case $a \neq b$, $\alpha \neq \beta$. By a straightforward computation,

$$\frac{f(a, b, \alpha) - f(a, b, \beta)}{|f(a, b, \alpha) - f(a, b, \beta)|} = \frac{a - b}{|a - b|} \text{sign}(\beta - \alpha).$$

Therefore, $v_{f(a, b, \alpha)f(a, b, \beta)} = v_{ab}$ and f satisfies (5), (6) as required.

Now, we prove continuity of f . For all $a, b \in \mathbb{E}$, $\gamma \in [0, 1]$, it follows that $|a - f(a, b, \gamma)| + |f(a, b, \gamma) - b| = |a - b|$. By the triangle inequality, the point $f(a, b, \gamma)$ belongs to the segment joining a and b . Thus, for every open ball $B \subset \mathbb{E}$, $f(B \times B \times [0, 1]) = B$. This proves the convexity of all open balls with respect to $\text{Arc } f$, as well as the continuity of f for $a = b$. If $a \neq b$, the continuity of f is obvious.

Example 3 (a solution equivalent to a non-linear connection). Consider Example 2 and suppose moreover that v is a diffeomorphism of class C^∞ . Since for $a = b$ some of the second partial derivatives of (16) do not exist, f is of class C^1 . Let $r : \mathbb{R} \rightarrow \mathbb{R}$ be a mapping defined by

$$\frac{\partial^2 v}{\partial t^2} = r \frac{\partial v}{\partial t},$$

where $t = \text{id}_{\mathbb{R}}$, and $s : \mathbb{E} \times \mathbb{E} \rightarrow \mathbb{E}$ is defined by

$$s(a, b) = \begin{cases} r((ab)/|b|) |b| b & \text{for } b \neq 0 \\ 0 & \text{for } b = 0. \end{cases}$$

Then all geodesics $g \in \text{Geo } f$ are solutions of the differential equation

$$\frac{\partial^2 g}{\partial t^2} + s\left(g, \frac{\partial g}{\partial t}\right) = 0.$$

Since for each $\lambda \in \mathbb{R}$, $a, b \in \mathbb{E}$, we have $s(a, \lambda b) = \lambda^2 s(a, b)$, these geodesics can be interpreted as geodesics of a non-linear, homogeneous connection (see e.g. [7]). In some special cases, e.g. for v defined by

$$v : \mathbb{R} \ni \alpha \rightarrow \int_0^\alpha \exp\left(\frac{\beta^2}{2}\right) d\beta \in \mathbb{R},$$

this connection is linear and smooth. If $v = \text{id}_{\mathbb{R}}$ we get the situation described by Example 1.

Example 4 (a solution which does not represent a connection).

Let $w : \mathbb{R} \rightarrow \mathbb{R}$ be the Weierstrass's function; w is continuous and bounded, but it is not differentiable at any point (see [2]). Since w is bounded, there exists $\kappa \in (0, \infty)$ such that for each $\alpha \in \mathbb{R}$, $-\kappa < w(\alpha) < \kappa$. Then by the continuity of w , the function

$$\mathbb{R} \ni \alpha \rightarrow \int_0^\alpha (w(\beta) + \kappa) d\beta \in \mathbb{R}, \quad (17)$$

is differentiable on \mathbb{R} , and its derivative is positive. Since the indefinite integral of the Weierstrass's function is a bounded function on \mathbb{R} , it follows that (17) is a surjection. Hence, (17) is a diffeomorphism of class C^1 .

Let v^{-1} in Example 2 be the mapping (17). Then by (8) and (16), the geodesic $g \in \text{Geo } f$ is not twice differentiable at any point of its domain. Therefore, g cannot be a solution of a second order differential equation. Consequently, there is neither a linear nor non-linear connection whose geodesic is g .

Acknowledgments

The author is grateful to professor Demeter Krupka for his constant attention to this work. He also thanks Jaroslav Štefánek and Michal Marvan for useful discussions. This work has been supported from the Project VS 96003 "Global Analysis" by the Czech Ministry of Education, Youth, and Sports.

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Lubomír Klapka
Department of Mathematics and Computer Science
Silesian University at Opava
Bezručovo nám. 13
746 01 Opava 1, Czech Republic